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## Equivalent Forms of the Axiom of Choice

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EQUIVALENT FORMS OF THE AXIOM OF CHOICE

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A Thesis

Presented to

The Faculty of the Department of Mathematics  
The College of William and Mary in Virginia

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In Partial Fulfillment  
Of the Requirements for the Degree of  
Master of Arts

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By

Leon Francis Sagan

August 1964

APPROVAL SHEET

This thesis is submitted in partial fulfillment of  
the requirements for the degree of  
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## ABSTRACT

The validity of the axiom of choice is constantly found to be a basis for disagreement among mathematicians. The purpose of this study is not to assert the validity or the lack of validity of the axiom of choice, but rather to demonstrate the equivalence of this axiom with other propositions from various branches of mathematics.

The study first devotes itself to establishing definitions, explaining notation and citing various well-known theorems and/or propositions which are necessary for the reading of the paper.

In asserting and proving three theorems, each of which composes a separate chapter, the axiom of choice is shown to be equivalent to various theorems and propositions in set theory and topology. The fourth chapter, in particular, discusses the equivalence of the axiom of choice and the law of trichotomy for cardinal numbers.

The study concludes with a brief discussion of the role of the axiom of choice in mathematics.

## **EQUIVALENT FORMS OF THE AXIOM OF CHOICE**

## INTRODUCTION

In 1904, Zermelo formulated a principle commonly referred to as the axiom of choice. The axiom of choice asserts that for every non-empty set  $X$ , there exists a function  $f$  on the collection of non-empty subsets,  $A$ , of  $X$  into  $X$ , such that for every  $A$ , the image of  $A$  under  $f$  is in  $A$ . The question of the validity of this principle gave rise to a controversy which divided mathematicians. Even today, although we find applications of the axiom of choice to problems of almost every domain of mathematics, there is at times skepticism concerning its use.

Assuredly the principle behind the axiom of choice was used many times before its actual formulation. Following the suggestion of Erhard Schmidt, Zermelo explicitly stated the principle of choice in 1904 and used it as the basis for his proof of the well-ordering theorem and again in 1908 for his second proof.

The axiom of choice does not assert that we can designate one element in every non-void set, nor that we can choose a particular element from every set. Actually the axiom merely asserts the existence of such an element. Here, then, we observe the basis of controversy surrounding the axiom of choice. It is the assertion of existence rather than the actual construction of the set which is the cause of many intuitionists regarding the principle as inadmissible



or lacking importance.

Despite the controversy surrounding the axiom of choice the results obtained from it have been indisputably valuable in the modern development of mathematics.

In chapter I the reader will be introduced to the notation, symbolism and definitions used throughout the paper.

In chapter II the axiom of choice is shown to be equivalent to propositions from set theory.

Chapter III discusses the axiom of choice and the Tychonoff Theorem.

In chapter IV we find that the law of trichotomy for cardinal numbers and the axiom of choice are equivalent.

In chapter V the reader will find a few applications of the axiom of choice.

## CHAPTER I

### PRELIMINARIES

This entire chapter contains definitions with which the reader should be familiar prior to reading the succeeding chapters.

If we have a function  $f$  which maps  $X$  into  $Y$ , then by  $f(x)$ , where  $x$  is in  $X$ , we mean the image of  $x$  under  $f$ . For every  $y$  contained in the range of  $f$  we define  $f^{-1}(y)$  to be the set of all  $x$  in  $X$  such that  $f(x)=y$ . If for every  $A$  open in  $X$ ,  $f(A)$  is open in  $Y$ , then  $f$  is said to be an open mapping.

A relation can be defined as a set of ordered pairs. If  $R$  is a relation we write  $xRy$  and  $(x,y) \in R$  interchangeably, and we say  $x$  is  $R$ -related to  $y$  if and only if  $xRy$ . The domain of a relation  $R$  is the set of all first coordinates of members of  $R$ , and its range is the set of all second coordinates. One of the simplest relations is the set of all pairs  $(x,y)$  such that  $x$  is a member of some fixed set  $A$  and  $y$  is a member of some fixed set  $B$ . This relation is the Cartesian product of  $A$  and  $B$ .

If  $\{X_j\}_{j=1}^n$  is a collection of sets, the product space, denoted by  $\prod_{j=1}^n X_j$  is defined as the set of all  $n$ -tuples of numbers  $\{x_j\}_{j=1}^n$  where for every  $j$  it is true that  $x_j$  is an element of  $X_j$ . To extend the concept of product spaces, let  $\{X_a\}_{a \in A}$  be a collection of sets indexed by  $A$ . Then the Cartesian product  $\prod_{a \in A} X_a$  of the sets of this collection is defined to be the set of all mappings  $f$ , defined on  $A$  into  $\bigcup_{a \in A} X_a$  such that  $f(a)$  is an element of the set  $X_a$  for all " $a$ " in  $A$ .

Given an arbitrary non-void set  $S$  and a relation  $R$  on the elements of  $S$  then we say the relation  $R$  partially-orders  $S$  if and only if

1) if  $a$  and  $b$  are in  $S$  such that  $aRb$  then not  $bRa$  and 2) if  $a$ ,  $b$  and  $c$  are in  $S$  such that  $aRb$  and  $bRc$  then  $aRc$ .

$R$  is said to simply-order  $S$  if  $R$  partially-orders  $S$  and if  $aRb$  or  $bRa$  is true for every  $a$  and  $b$  in  $S$ . If  $S$  is partially-ordered by  $R$  and  $S'$  is a subset of  $S$  which is simply-ordered by  $R$ , then  $S'$  is called a chain in  $S$ .

If  $R$  is a relation which simply-orders  $S$  then  $R$  is said to well-order  $S$  if every non-empty subset of  $S$  has a first or minimal element - i.e. for every non-empty subset  $A$  of  $S$  there exists an element  $x$  in  $A$  such that  $xRy$  for all  $y$  not equal to  $x$  in  $A$ .

In our discussion of chains in Chapter II we sometimes refer to a chain in some arbitrary set <sup>of sets</sup>  $S$ . By this we assume that we have partially-ordered  $S$  by inclusion,  $\subset$ , and then the definition of a chain in  $S$  conforms with our above definition.

By  $A \subseteq B$  we mean  $A$  is a subset of  $B$  and  $A < B$  means  $A$  is a proper subset of  $B$ .

If  $N'$  is a chain in some set  $S$  then by union of  $N'$  or  $\bigcup N'$  we mean  $\bigcup_{N \in N'} N$  (similarly for intersection).

If  $A$  and  $B$  are arbitrary sets, by  $A-B$  we mean the set of all  $x$  such that  $x$  is in  $A$  but not in  $B$ .

Suppose that  $S$  is a set partially-ordered by  $\alpha$  then  $x$  is a maximal element of  $S$  if there does not exist  $x'$  in  $S$  such that  $x \alpha x'$ .

A non-empty set of sets  $\underline{A}$  is said to be of finite character if and only if 1) every finite subset of a member of  $\underline{A}$  is also a member

of  $\underline{A}$  and 2) if every finite subset of a set is a member of  $\underline{A}$  then the set is also a member of  $\underline{A}$ .

Let  $s$  be a set and  $\sigma$  a collection of subsets of  $S$ . Then  $\sigma$  is said to generate the collection  $T$  of subsets of  $S$  defined as follows: A subset  $K$  of  $S$  is an element of  $T$  if and only if  $K$  is the union of a collection of elements of  $\sigma$ . The collection  $\sigma$  is said to be a basis for the collection  $T$  which it generates.

Let  $S$  be a non-empty set and  $T$  a class of subsets of  $S$ . Then  $S$  is a topological space, with topology  $T$ , if and only if 1) for any collection  $G^*$  of sets in  $T$  we have  $\bigcup_{G \in G^*} G$  an element of  $T$  and 2) for any finite collection  $G'$  of sets in  $T$  we have  $\bigcap_{G \in G'} G$  an element of  $T$ .

Hall and Spencer [2]<sup>1</sup>, p. 54 prove the following: Let  $S$  be a set and  $T$  a collection of subsets of  $S$  generated by  $\sigma$ . Then,  $S$  is a topological space with the topology  $T$  if and only if the following hold: (1) Given  $p$  an element of  $S$ , there exists a set  $U$  in  $\sigma$  such that  $p$  is in  $U$ . (2) Given  $U$  and  $V$  elements of  $\sigma$  and any point  $p$  in  $U \cap V$  then there exists an element  $W$  of  $\sigma$  such that  $p$  is in  $W$  and  $W \subset U \cap V$ .

Let  $\underline{F}$  be a non-empty family of sets.  $\underline{F}$  is said to possess the finite intersection property if for every finite subcollection  $F^*$  of  $\underline{F}$  we have  $\bigcap_{F \in F^*} F \neq \emptyset$ .  $\underline{F}$  is said to possess the intersection property if  $\bigcap_{F \in \underline{F}} F \neq \emptyset$ .

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<sup>1</sup>The numbers in brackets refer to the bibliography.

The following discussion on cardinal numbers is taken from Sierpinski [9], pp.132-134.

Two sets are said to be equivalent or to be of the same power if there exists a one-to-one correspondence between the elements of these sets. It is apparent that the reflexive, symmetrical and transitive laws hold for the relation of equivalence.

Suppose now that we have a family of sets. Let us divide the sets into classes, assigning two sets to the same class if and only if they are equivalent. The classes are called equivalence classes and we call them cardinal numbers. Cardinal numbers relating to the same class are regarded as equal. Cardinal numbers are usually regarded as classes of all sets equivalent to (of the same power as) a certain given set.

If  $M$  is a given set and  $m$  the cardinal number serving to denote the class of equivalent sets to which  $M$  belongs, then we shall say that to the set  $M$  corresponds the cardinal number  $m$ , or that the set  $M$  is of power  $m$ , and we shall write  $\bar{M}=m$  and  $m=\text{card } M$ .

Since for finite sets the concept of equivalent sets is synonymous with the concept of sets with equal number of elements, it is simplest and most convenient to adopt natural numbers as the respective symbols of cardinal numbers. As the cardinal number of the empty set we adopt the number 0.

The cardinal number corresponding to denumerable sets - i.e. sets which can be placed in a one-to-one correspondence with the set of positive integers - is denoted, after G. Cantor, by the symbol  $\aleph_0$ .

Let  $A$  and  $B$  be two given sets. We write  $\bar{A} < \bar{B}$  if the set  $A$  is equivalent to (of the same power as) a certain subset of the set  $B$ ,

but the set B is not equivalent to any subset of the set A.

The Law of Trichotomy for cardinal numbers states that if  $A_1$  and  $A_2$  are two arbitrary sets, then always one and only one of the formulae  $\bar{A}_1 = \bar{A}_2$ ,  $\bar{A}_1 < \bar{A}_2$ ,  $\bar{A}_1 > \bar{A}_2$  holds.

A function  $f$  is said to be a similar mapping of the partially-ordered set A onto the partially-ordered set B if  $f$  is a one-to-one mapping of A onto B and if  $f$  preserves the ordering.

Let R be a relation defined on the elements of a set A. Each subset  $A_1$  of A such that if "a" is in  $A_1$  and x is in A with  $xRa$ , then x is in  $A_1$ , is called a segment of the set A.

Suppose now that A is well-ordered by R. Let  $A_1$  be a segment of A. The set  $A - A_1$  is well-ordered and has a unique first element, say "a". Define  $A_a$  as the set of all x in A such that  $xRa$  is true. Clearly  $A_1 = A_a$ .

We define the power of every well-ordered set as an aleph, and conversely every aleph is the power of a certain well-ordered set.

If M is a well-ordered set and  $\aleph_M$  the power of M then we designate the power of the class of all subsets of M by  $2^{\aleph_M}$ .

Halmos [3], pp.19,30-33 introduces the concept of measure by means of set functions and rings. Halmos defines a ring as a non-empty class R of sets such that if E and F are in R then  $E \cup F$  and  $E - F$  are in R. A set function is a function whose domain of definition is a class of sets. An extended real-valued set function  $\mu$  defined on a class  $\mathcal{E}$  of sets is additive whenever E and F are disjoint elements of  $\mathcal{E}$  such that their union is also in  $\mathcal{E}$  then  $\mu(E \cup F) = \mu(E) + \mu(F)$ . Further  $\mu$  is countably additive or  $\sigma$ -additive if for every disjoint sequence  $\{E_n\}$  of sets in  $\mathcal{E}$  whose union is also in  $\mathcal{E}$ , we have  $\mu(\bigcup_{n=1}^{\infty} E_n) =$

$\sum_{n=1}^{\infty} \mu(E_n)$ . A measure is an extended real valued, non-negative and countably additive set function defined on a ring. The measure of the empty set is defined as 0.

Now a  $\sigma$ -ring is a non-empty class  $\underline{S}$  of sets such that 1) if  $E$  and  $F$  are elements of  $\underline{S}$  then  $E \cup F$  is in  $\underline{S}$  and 2) if  $E_i$  is an element of  $\underline{S}$  for  $i=1,2,3,\dots$ , then  $\bigcup_{i=1}^{\infty} E_i$  is in  $\underline{S}$ .

Suppose, now, that we have a  $\sigma$ -ring  $\underline{S}$  and a measure defined on  $\underline{S}$ . It is customary to call a subset  $E$  of a set measurable if and only if it belongs to the  $\sigma$ -ring  $\underline{S}$ .

## CHAPTER II

### SET THEORETIC EQUIVALENCES

We prove the equivalence of the axiom of choice with several propositions from set theory by stating and proving the following theorem:

Theorem 1 The following are equivalent:

- (1) The Hausdorff Maximal Principle: Every chain in a non-empty family of sets  $\underline{A}$  is a subset of some maximal chain in  $\underline{A}$ .
- (2) Every family of sets containing a chain has at least one maximal chain.
- (3) The Maximality Principle: <sup>If</sup> For every chain  $N$  in a non-empty family,  $\underline{S}$ , of sets, there exists a set  $A$  in  $\underline{S}$  containing the union of  $N$  then there exists a maximal set of  $\underline{S}$ .
- (4) The Tukey Lemma: If  $\underline{A}$  is a non-empty family of sets with finite character then there exists a maximal member of  $\underline{A}$ .
- (5) Kuratowski's Lemma: If  $B$  is a chain in a partially-ordered set  $A$ , then there exists a maximal chain  $M$  having  $B$  as a subset.
- (6) Zorn's Lemma: If every chain  $B$  in a partially-ordered set  $A$  has an upper bound, then  $A$  has a maximal element.
- (7) The Well-Ordering Theorem: Every non-empty set can be well-ordered.
- (8) Given a relation  $R$  on an arbitrary set  $X$  then there exists a function  $f$  such that  $f$  is a subset of  $R$  and the domain of  $R$  is equal to the domain of  $f$ .
- (9) Given a collection of non-empty sets  $\{X_a\}_{a \in A}$  the product space of the collection is non-void.



- (10) If  $\underline{A}$  is a family of non-empty pairwise disjoint sets then there exists a set  $C$  such that for every  $A$  in  $\underline{A}$  the intersection of  $C$  with  $A$  is a unit set.
- (11) Zermelo's Proposition: If  $\underline{G}$  is a family of non-empty sets then there exists a set  $B$  of ordered pairs  $(G, x)$  where  $G$  is an element of  $\underline{G}$  and  $x$  is an element of  $G$  and further for every  $G$  in  $\underline{G}$  it is true that there exists a unique ordered pair  $(G, y)$  in  $B$ .
- (12) The Axiom of Choice: For every non-empty set  $A$  there exists a function  $f$  defined on the non-empty subsets of  $A$  such that for every non-empty subset  $B$  of  $A$   $f(B)$  is an element of  $B$ .

Proof:

(1)  $\rightarrow$  (2)

The implication of (2) by the Hausdorff Maximal Principle, (1), is trivial. To show this implication let  $\underline{A}$  be a family of sets containing a chain. By our hypothesis we have the existence of a maximal chain in  $\underline{A}$  which contains the given chain. Thus, our conclusion is immediate.

(2)  $\rightarrow$  (3)

We now show that (3), the maximality principle, is implied by (2). To do this let  $\underline{S}$  be a non-empty family of sets; let  $N$  be a chain in  $\underline{S}$ . By (2) there exists at least one maximal chain  $\underline{M}$  in  $\underline{S}$ . Our hypothesis then states there exists a set  $A$  in  $\underline{S}$  such that every set composing  $\underline{M}$  is a subset of  $A$ . Define  $X=A$ . We assert  $X$  is maximal in  $\underline{S}$ . Suppose not. That is, suppose there does exist a set  $B$  in  $\underline{S}$  such that  $X$  is a proper subset of  $B$ . Then the collection of sets  $M'$  which is composed of  $B$  and every set in  $\underline{M}$  is a chain in  $\underline{S}$ . This is evident for let  $m$  and  $m'$  be elements in  $M'$ . We have two possibilities: either both  $m$  and  $m'$  are elements of the chain  $\underline{M}$  (whence  $m < m'$  or  $m' < m$ ); or one of them, say  $m$ ,

is an element of  $\underline{M}$  and  $m' \in B$ . The latter possibility yields  $m \subset m'$  since every element of  $\underline{M}$  is a subset of  $A=X$  which is in turn a proper subset of  $B$ . Not only is  $M'$  now a chain but it properly contains  $\underline{M}$  which contradicts  $\underline{M}$  being maximal in  $\underline{S}$ . Thus  $X$  must be maximal.

(3)  $\rightarrow$  (4)

In showing (3), the maximality principle, implies (4), the Tukey Lemma, we use the property of finite character. The reader is referred back to page 5. To show this implication let  $\underline{A}$  be a family of sets with finite character and let  $N'$  be a chain in  $\underline{A}$ . Define set  $A$  to be the union of  $N'$ . Further, let us now assume  $F$  is a finite subset of  $A$ . We assert that  $F$  is a subset of some  $N$  in  $N'$ , whence by the first defining property for finite character we have  $F$  an element of  $\underline{A}$ . (The second defining property then yields  $A$  an element of  $\underline{A}$ .) Note  $N'$  is a chain which implies that for  $N_1$  and  $N_2$  elements of  $N'$  we have either  $N_1 \subset N_2$  or  $N_2 \subset N_1$ . Now  $F$  is a finite subset of  $A$  by assumption, thus there must exist  $\underline{G}$  a finite subfamily of  $N'$  such that  $F$  is itself contained in the union of all the  $G$ 's in  $\underline{G}$ . Since  $\underline{G}$  is finite we may assert  $\underline{G}$  has a largest member, call it  $G'$ , containing  $F$ . Now  $G'$  being an element of  $\underline{G}$  forces it to be an element of  $N'$ . Define  $N=G'$ . Thus for every  $F$ , finite, contained in  $A$  there exists a set  $N$ , an element of  $N'$ , which contains  $F$  as a subset and which is in turn a subset of  $\underline{A}$ .  $\underline{A}$  being of finite character by hypothesis then gives us  $A$  an element of  $\underline{A}$ . Then  $\underline{A}$  satisfies the hypothesis of the maximality principle, hence  $\underline{A}$  has a maximal member.

(4)  $\rightarrow$  (5)

In order to show (4) implies (5) let  $B$  be a chain in a partially-ordered set  $A$  and define  $\underline{B}$  to be the family of all subsets of  $A$  whose

union with  $B$  is a chain. Notice  $\underline{B}$  is not void since  $B$  is by definition in  $\underline{B}$ .

We now show  $\underline{B}$  is of finite character. To this end let  $C'$  be an element of  $\underline{B}$  and let  $C$  be a finite subset of  $C'$ . Now  $C'$  union  $B$  is a chain - i.e. for every  $m$  and  $m'$  contained in their union either  $m < m'$  or  $m' < m$ . For every  $n$  and  $n'$  contained in the union of  $C$  and  $B$  we have  $n$  and  $n'$  contained in the union of  $C'$  and  $B$ ; thus we have either  $n < n'$  or  $n' < n$ . We now have  $C$  belonging to  $\underline{B}$ . What we have shown then is that for every finite subset of a member of  $\underline{B}$  it is itself a member of  $\underline{B}$ . The second defining property for finite character obviously holds.

It is evident that  $\underline{B}$  is of finite character, hence by Tukey's Lemma there exists a maximal set  $B'$  in  $\underline{B}$ . Now the union of  $B'$  with  $B$  is a chain, call it  $B^*$ . We assert  $B^*$  is maximal. If there exists a chain  $R$  containing  $B^*$  as a proper subset it would have  $B$  and  $B'$  as proper subsets. Then  $R$  would belong to  $\underline{B}$  which contradicts  $B'$  being a maximal set in  $\underline{B}$ .

(5)  $\rightarrow$  (6)

We wish now to show that (5), Kuratowski's Lemma, implies proposition (6), known as Zorn's Lemma. To this end let  $B$  be a chain in the partially-ordered set  $A$ . Then by (5) we have the existence of a maximal chain  $M$  of  $A$  such that  $B$  is contained in  $M$ . Our hypothesis then gives us an upper bound, say  $x$ , of  $M$ . We assert  $x$  is a maximal element of  $A$ . Suppose that  $x$  precedes  $y$  (according to the partial-ordering of  $A$ ) in  $A$ . Then every  $m$  belonging to  $M$  precedes  $y$ . Now the set composed of all the elements of  $M$  and  $y$  is easily verified to be a chain in  $A$ . Further,  $M$  is now a proper subset of this chain which contradicts  $M$  being maximal. Therefore there is no such  $y$ .

(6)  $\rightarrow$  (7)

One of the most frequently used principles which will be seen to be equivalent to the axiom of choice when the proof of Theorem 1 is completed is the Well-Ordering Theorem. Using Zorn's Lemma, (6), we now prove the Well-Ordering Theorem, (7). The proof is due to Stoll [10], pp 116-118. From the definitions of simply-ordered sets and well-ordered sets it is clear that to insure a relation  $R$  well-orders a set  $A$  it is sufficient that  $R$  be antisymmetric and that in each non-empty subset  $A_1$  there exists an element  $a_1$  such that  $a_1 R b$  for every  $b$  in  $A_1$  where  $b$  is different from  $a_1$ .

Let  $X$  be any set and let  $\underline{S}$  be the set of all ordered pairs  $(A, R)$  where  $A$  is a subset of  $X$  and  $R$  well-orders  $A$ . If  $(A_1, R_1)$  and  $(A_2, R_2)$  are members of  $\underline{S}$  define  $(A_1, R_1) < (A_2, R_2)$  if and only if a)  $A_1$  is a proper subset of  $A_2$ , b)  $R_1$  is a proper subset of  $R_2$  and c) for  $a_1$  an element of  $A_1$  and  $a_2$  an element of  $A_2 - A_1$ , then  $(a_1, a_2)$  is part of the ordering by  $R_2$ . That is to say we require  $A_1$  to be a proper subset of  $A_2$ , that the ordering of  $A_2$  be an extension of that of  $A_1$ , and every element of  $A_1$  precedes every element of  $A_2 - A_1$ . It is immediate that  $<$  partially-orders  $\underline{S}$ .

We prove next that  $(\underline{S}, <)$  satisfies the hypothesis of Zorn's lemma; that is, every chain in  $\underline{S}$  has an upper bound in  $\underline{S}$ . For a chain  $C$  which is a subset of  $\underline{S}$  we propose as an upper bound the ordered pair  $(A', R')$  where  $A'$  is the union of all the  $A$ 's in the ordered pairs  $(A, R)$  which belong to  $C$  and  $R'$  is the union of all of the  $R$ 's for the same ordered pairs. Clearly the question is whether  $(A', R')$  belongs to  $\underline{S}$ . To show this we prove  $(A', R')$  satisfies the conditions cited above.  $R'$  is antisymmetric. We let  $B$  be a non-void subset of  $A'$  and assert that

there exists  $b_0$  in  $B$  such that  $(b_0, b)$  is part of the ordering of  $R'$  for every  $b$  in  $B$ . For such a  $B$  there then exists  $(A_1, R_1)$  belonging to  $C$  such that the intersection of  $B$  with  $A_1$  is not void. Since  $R_1$  is a well-ordering of  $A_1$ , there exists  $b_0$  an element of this intersection such that  $(b_0, b)$  is part of the ordering of  $R_1$  for all  $b$  not equal to  $b_0$  in  $B \cap A_1$ . More generally, for every  $b$  in  $B$  there exists  $R$  with  $(A, R)$  an element of  $C$  such that  $(b_0, b)$  is part of the ordering of  $R$  - i.e. being an element of  $B$ ,  $b$  is contained in some  $A$  for which  $R$  well-orders  $A$  and  $(A, R)$  is an ordered pair in  $C$ . Indeed, given  $b$  an element of  $B$ , there exists an element  $(A, R)$  of  $C$  with  $b$  an element of  $A$ . If  $A$  is a subset of  $A_1$  then  $(b_0, b)$  is part of the ordering of  $R_1$ , otherwise  $A$  is a superset of  $A_1$  and hence  $R$  contains  $R_1$ . Then  $(b_0, b)$  belongs to  $R$  and so  $(b_0, b)$  belongs to  $R'$  as desired.

Now we may infer the existence of a maximal element  $(A, R)$  of  $\underline{S}$ . The proof will be complete if it can be shown that  $A=X$ . To this end assume the contrary, i.e. assume there exists  $x$  in  $X-A$ . Adjoin  $x$  to  $A$  and extend the ordering of  $A$  to one of  $A \cup \{x\}$  by defining  $x$  to be greater than each element of  $A$ . This yields the ordered pair  $(A', R')$  where  $A'$  is the union of  $A$  and  $\{x\}$  and  $R' = R \cup \{(a, x) \text{ for } a \in A\}$  united with the set of all ordered pairs  $(a, x)$  for 'a' ranging over the elements of  $A$ . Then  $R'$  well-orders  $A'$  and hence  $(A', R')$  is an element of  $\underline{S}$ . Moreover,  $(A, R) < (A', R')$  which contradicts  $(A, R)$  being a maximal element. Hence  $X=A$ .

(7)  $\rightarrow$  (8)

The next proof in our theorem is the implication of (8) by the well-ordering theorem. For this proof let  $R$  be a relation on an arbitrary non-empty set  $X$  and let  $x$  be an element of the domain of  $R$ . By hypothesis  $X$  can be well-ordered. Define  $R_x$  as the set of all  $y$  in

$X$  such that  $(x, y)$  is an ordered pair in  $R$ .  $R_x$  being a subset of the well-ordered set  $X$  is itself well-ordered. Let  $g(x)$  be the first element in  $R_x$ . Obviously then  $(x, g(x))$  is an element of  $R$ . If we now define  $f(x)$  to be the set of all ordered pairs  $(x, g(x))$  where  $x$  ranges over the domain of  $R$  we have  $f$  as a function satisfying the desired properties.

(8)  $\rightarrow$  (9)

We now turn our attention to product spaces in order to show (8) implies (9). Let  $\{X_a\}_{a \in A}$  be a collection of non-empty sets. For every "a" in  $A$  define the relation  $R$  to be the set of all ordered pairs  $(X_a, x_a)$  where  $x_a$  is an element of  $X_a$ . (8) then asserts the existence of a function  $f$  such that  $f$  is a subset of  $R$  and the domain of  $f$  is the same as the domain of  $R$ . We have then for every  $X_a$  an element  $x_a$  of  $X_a$  with  $(X_a, x_a)$  an ordered pair in the function  $f$ . Further, it is evident that  $x_a$  is unique for by the definition of a function we cannot have  $x_a$  and  $x'_a$  both elements of  $X_a$  such that  $(X_a, x_a)$  and  $(X_a, x'_a)$  are ordered pairs of the function. We now define  $\Delta$  to be the set of all  $x_a$  such that  $x_a$  is in the range of  $f$  and note that  $\{x_a\}_{x_a \in \Delta}$  is a point of the product space - whence  $\prod_{a \in A} X_a$  is non-empty.

(9)  $\rightarrow$  (10)

The following demonstrates how (9) implies (10): Let  $\{A_r\}_{r \in R}$  be a family of non-empty pairwise disjoint sets. By hypothesis then  $\prod_{r \in R} A_r$  is non-void. Therefore let  $\{a_r\}_{r \in R}$  be a point in the product space. Note that  $a_{r_1}$  is an element of  $A_{r_1}$  for  $r_1$  in  $R$ . Define  $C$  to be the set of coordinates of  $\{a_r\}_{r \in R}$ . Obviously, then, for every  $r$  in  $R$  it is true that the intersection of  $C$  with  $A_r$  is the unit set  $\{a_r\}$ .

(10)  $\rightarrow$  (11)

The next proof in our theorem is the implication of Zermelo's proposition by (10). To prove the implication let  $\underline{G}$  be a family of non-empty sets. For  $G$  contained in  $\underline{G}$  define  $S_G$  as the set of all ordered pairs  $(G, x)$  where  $x$  is an element of  $G$ .  $S_G$  is non-void since each  $G$  is itself non-void. Notice now that the family of sets  $\underline{S} = \{S_G\}_{G \in \underline{G}}$  is a family of non-empty sets. Further for every  $G$  and  $G'$  in  $\underline{G}$  we have  $S_G \cap S_{G'} = \phi$  if  $G \neq G'$ .  $\underline{S}$  satisfies the hypothesis to (10). Thus there exists a set  $B$  such that for every  $S$  in  $\underline{S}$  the intersection of  $B$  with  $S$  is a unit set.  $B$ , trivially, satisfies the conclusions to (11).

(11)  $\longrightarrow$  (12)

We now show (11) implies (12). The proof is trivial, for let  $A$  be a non-empty set and define  $\underline{A}$  as the set of all non-empty subsets of  $A$ . Applying (11) to  $\underline{A}$  there then exists a set  $B$  of ordered pairs  $(A_i, x)$  where  $A_i$  is in  $\underline{A}$  and  $x$  is an element of  $A_i$  and further for every  $A_j$  in  $\underline{A}$  there is a unique ordered pair  $(A_j, y)$  in  $B$ . Now let  $C$  be a non-empty subset of  $A$ . Then there exists the unique ordered pair  $(C, x_C)$  in  $B$  where  $x_C$  is in  $C$ . For every  $C$  in  $\underline{A}$  define  $f(C) = x_C$  where  $(C, x_C)$  is in  $B$ .  $x_C$  is unique and  $x_C$  is in  $C$  whence  $f(C)$  is a function and  $f(C)$  is an element of  $C$ .

(12)  $\longrightarrow$  (1)

Prior to proving (12) implies (1) we prove the following lemma:

Let  $E$  be a non-void partially-ordered set such that every chain included in  $E$  has a least upper bound in  $E$ . If  $f$  maps  $E$  into  $E$  has the property  $f(x) \geq x$  for every  $x$  in  $E$  then there exists at least one  $x$  in  $E$  such that  $f(x) = x$ .

The proof is taken from Stoll [10], pp 113-116. As a matter of definition let  $A$  be a subset of  $E$  and ' $m$ ' a fixed element of  $E$ . We say  $A$  is admissible

relative to ' $m$ ' if and only if 1) ' $m$ ' is an element of  $A$ , 2)  $f(A)$  is a subset of  $A$  and 3) if  $F$  is a chain in  $A$  then the least upper bound of  $F$ ,  $\text{lub } F$ , is an element of  $A$ .

From the above definition  $E$  is clearly admissible to say  $m$ . Let  $\underline{M}$  be the set of all  $M'$  in  $E$  which are admissible (it will be understood, although not explicitly stated, admissible relative to  $m$ ) and define  $M$  as the intersection of all the  $M'$  in  $\underline{M}$ . For every chain  $C$  in  $M$ ,  $C$  is a subset of each  $M'$  which is admissible and hence the  $\text{lub}$  of the chain  $C$  is contained in each  $M'$ . This, then, says that the  $\text{lub } C$  is contained in  $M$ . Thus, we have  $M$  admissible. Note that this is the smallest admissible set in  $E$ . Therefore if  $M_0$  is a subset of  $M$  such that  $M_0$  is admissible we have  $M_0 = M$ .

Let  $A$  be defined as the set of all  $x$  in  $M$  such that  $x \geq m$ . We prove that  $A$  is admissible.

I (1)' Since  $m$  is an element of  $M$  and  $m \geq m$ ,  $m$  must be an element of  $A$ .

(2)' Let  $x$  be an element of  $A$ . Then  $x$  is an element of  $M$  which is admissible - whence  $f(x)$  is an element of  $M$ . Since for  $x$  in  $A$  we have  $x \geq m$  and  $f(x) \geq x$ . This allows us to assert  $f(x)$  is an element of  $A$ .

(3)' Let  $w = \text{lub } F$  where  $F$  is a chain in  $A$ . Since  $A$  is a subset of  $M$  we have  $F$  a subset of  $M$  and hence  $w$  an element of  $M$  by 3).  $w$  is the  $\text{lub } F$  which says for every  $x$  in  $F$  we have  $x \leq w$ . Also,  $F$  is a subset of  $A$  which says for every  $x$  in  $F$ ,  $m \leq x$ ; hence  $w$  is an element of  $A$ .

Thus  $A$  is admissible and by the above remarks  $A = M$ .

We will say an element  $x$  of  $E$  has property  $P$ , i.e.  $P(x)$ , if for



y an element of  $M$  with the property  $y < x$  then  $f(y) \leq x$  is true.

II. If  $x$  is an element of  $M$  and  $P(x)$  is true then the set  $B$ , defined as the set of all  $z$  in  $M$  such that  $z \leq x$  or  $z \geq f(x)$  is admissible. This follows from the following:

(1)"  $m$  is an element of  $M$  and  $m \leq x$ . By I  $m \leq x$  is true for all  $x$  in  $M$ .

Thus  $m$  is an element of  $B$ .

(2)" Let  $z$  be an element of  $B$ . We show  $f(z)$  is an element of  $B$ . As

in (2)'  $f(z)$  is an element of  $M$ . Also,  $z$  is an element of  $B$

which says  $z \leq x$  or  $z \geq f(x)$ . If  $z = x$  then  $f(z) = f(x)$  so that

$f(z) \geq f(x)$ . Hence  $f(z)$  is an element of  $B$ . Now if  $z < x$ ,

$P(x)$  tells us  $f(z) \leq x$  and  $f(z)$  belongs to  $B$ . If  $z \geq f(x)$ , it

is true that  $f(z) \geq z \geq f(x)$  and again  $f(z)$  is an element of  $B$ .

(3)" Let  $w = \text{lub } F$  for  $F$  a chain in  $B$ . As in (3)'  $w$  is an element of  $M$ .

Also, for every  $z$  belonging to  $F$  either  $z \leq x$  or  $z \geq f(x)$ . If

$z \leq x$  for every  $z$  in  $F$  we have  $x$  as an upper bound for  $F$  and

hence  $w \leq x$  so that  $w$  belongs to  $B$ . Otherwise there exists a

$z$  in  $F$  such that  $z \geq f(x)$  which implies  $w \geq z \geq f(x)$  and

again  $w$  is in  $B$ .

Thus  $B$  is admissible and hence  $B = M$ .

III. Every element of  $M$  has property  $P$ . We prove that the set  $C$  composed of all the elements  $x$  in  $M$  such that  $P(x)$  is true is admissible - whence  $C$  being a subset of  $M$  implies  $C = M$ .

(1)"  $m$  belongs to  $M$  such that  $m$  is a least element of  $M$ . Thus there

does not exist a  $z$  in  $M$  such that  $z < m$ . Hence  $m$  satisfies  $P$

vacuously and hence belongs to  $C$ .

(2)" Let  $x$  belong to  $C$ . We show  $f(x)$  belongs to  $C$ . As in (2)'  $f(x)$

belongs to  $M$ . We prove  $f(x)$  has property  $P$ , i.e. if  $y$  is in  $M$  and  $y < f(x)$  then  $f(y) \leq f(x)$ . Applying II to  $x$  we have  $y \leq x$  or  $y \geq f(x)$  but  $y \geq f(x)$  cannot hold. Thus  $y \leq x$ . If  $y < x$  we have  $f(y) \leq x$ . Now using property  $P$  for  $x$  and the fact that  $x \leq f(x)$  we have  $f(y) \leq f(x)$ . If  $y=x$  the same conclusion is immediate. Thus  $f(x)$  is in  $C$ .

(3)<sup>ii</sup> Let  $w = \text{lub } F$  where  $F$  is a chain in  $C$ . As in (3)<sup>i</sup>  $w$  is in  $M$ .

Thus it remains to show that  $P(w)$ , that is, for  $y$  in  $M$  and  $y < w$  we have  $f(y) \leq w$ . First we show that for such a  $y$  there exists a  $y'$  in  $F$  such that  $y \leq y'$ ; whence  $y'$  belonging to  $F$  would imply  $P(y')$ . If there did not exist a  $y'$  in  $F$  such that  $y \leq y'$ , then  $y \geq f(y') \geq y'$  is true for all  $y'$  in  $F$ . Then  $y$  is an upper bound for  $F$  whence  $y \geq w$  which is contrary to our assumption. Thus there does exist a  $y'$  in  $F$  such that  $y \leq y'$ . If  $y < y'$  is true then applying property  $P$  to  $y'$  we have  $f(y) \leq y' \leq w$ . If  $y=y'$  then  $P(y)$  and II imply either  $w \leq y$  or  $f(y) \leq w$ . The first possibility can be excluded and hence again  $f(y) \leq w$ . Thus  $C$  is admissible and  $C=M$ .

From II and III if  $x$  and  $y$  are elements of  $M$  then  $y \leq x$  or  $y \geq f(x) \geq x$  is true. Therefore  $M$  is simply-ordered. Let  $x_0 = \text{lub } M$ .  $M$  is admissible which says  $x_0$  is in  $M$  and  $f(x_0)$  is in  $M$ . Thus  $f(x_0) \leq x_0$  is true. But  $x_0 \leq f(x_0)$  is true by hypothesis. It follows that  $f(x_0) = x_0$ .

Now for the actual proof of (1) having (12) as our hypothesis we let  $\underline{A}$  be an arbitrary non-empty family of non-empty sets and assume (1) to be false. Partially-order the sets in  $\underline{A}$  by inclusion. Denial of (1) says there exists a chain  $C_0$  in  $\underline{A}$  which is not a subset of a maximal chain. Define  $\underline{C}$  as the set of all chains in  $\underline{A}$  containing the chain  $C_0$ .

Notice  $\underline{C}$  is non-void since  $C_0$  is itself in  $\underline{C}$ . Now, partially-order  $\underline{C}$  by inclusion and let  $\{T_a\}_{a \in R} = T'$  be a chain in  $\underline{C}$ . Observe that each  $T_a$  is itself a chain in  $\underline{A}$ . Define  $T$  to be the union of  $T'$ . We assert  $T$  is a chain in  $\underline{A}$ . If  $t_1$  and  $t_2$  are elements of  $T$  then there exists  $T_{a_1}$  and  $T_{a_2}$  in  $T'$  such that  $t_1$  is in  $T_{a_1}$  and  $t_2$  is in  $T_{a_2}$ . Being in a chain we have either  $T_{a_1} \subset T_{a_2}$  or  $T_{a_2} \subset T_{a_1}$ . Without loss of generality let  $T_{a_1} \subset T_{a_2}$ . Consequently, we have both  $t_1$  and  $t_2$  in the chain  $T_{a_2}$ , whence  $t_1 \subset t_2$  or  $t_2 \subset t_1$ . Thus  $T$  is a chain.  $T$  obviously contains  $C_0$ . Further it is evident that  $T$  is the least upper bound for the chain  $T'$ .

Now denial of (1) says that there exists no maximal element in  $\underline{C}$ . Let  $g$  be the function defined on the subsets of  $\underline{C}$  which is guaranteed by (12). For each  $C$  in  $\underline{C}$  define  $S(C)$  to be the set of all chains in  $\underline{C}$  which properly contain  $C$ . Now it is clear that for each  $C$  we have  $S(C)$  non-void for otherwise  $C$  is maximal in  $\underline{C}$ . Further, each  $S(C)$  is a subset of  $\underline{C}$ ; therefore define  $f(C) = g(S(C))$ . We then have  $f(C)$  contained in  $S(C)$  - i.e. for every  $C$  in  $\underline{C}$  it is true that  $C$  is properly contained in  $f(C)$ .

$\underline{C}$  and  $f$  satisfy the hypothesis to our previous lemma; however,  $C$  being a proper subset of  $f(C)$  for every  $C$  in  $\underline{C}$  is a contradiction to the conclusion of the lemma. Thus (1) must be true.

## CHAPTER III

### TOPOLOGICAL EQUIVALENCES

Before discussing the axiom of choice in relation to the Tychonoff Theorem we prove two lemmas.

Let  $\{X_a\}_{a \in A}$  be a collection of non-empty spaces. Define  $\sigma$  to be the collection of all subsets of the product space  $\prod_{a \in A} X_a$  that are of the form  $\prod_{a \in A} Y_a$  where for some finite subset  $E$  of the index set  $A$  we have  $Y_a$  an open set in  $X_a$  for all " $a$ " in  $E$  and  $Y_a = X_a$  for all " $a$ " in  $A - E$ . Notice in  $\sigma$  the elements have all but a finite number of the coordinates coming from some  $X_b$  in  $\{X_a\}_{a \in A}$  while a finite number have their coordinates coming from some open set in the remaining  $X$ 's.

Lemma 1':  $\sigma$  forms a basis for a topology of the product space  $\prod_{a \in A} X_a$ .

To show that  $\sigma$  is a basis we prove (i) given any point of the product space  $\prod_{a \in A} X_a$  then there exists an element of  $\sigma$  which contains this point and (ii) given any two sets in  $\sigma$  and some arbitrary point in their intersection then there is another set in  $\sigma$  containing this point which is in turn a subset in the intersection of the first two sets.

Using the notation above, if we let  $E$  be the empty set then we have the product space  $\prod_{a \in A} X_a$  being an element of  $\sigma$ . Thus, trivially (i) is true. In showing property (ii) we assert that if  $U$  and  $V$  are elements of  $\sigma$  then their intersection is itself an element of  $\sigma$ , which is a stronger result than needed. To show this let  $\prod_{a \in A} Y_a$  and  $\prod_{a \in A} Z_a$  be

elements of  $\sigma^-$ . By definition then there exist  $E$  and  $B$  finite subsets of  $A$  such that  $Y_b$  is an open subset of  $X_b$  for all  $b$  in  $B$  and  $Y_a = X_a$  for all " $a$ " in  $A-B$ . Likewise  $Z_e$  is an open subset of  $X_e$  for all  $e$  in  $E$  and  $Z_a = X_a$  for all " $a$ " in  $A-E$ . Consider now the intersection of  $\prod_{a \in A} Y_a$  and  $\prod_{a \in A} Z_a$ . This is obviously  $\prod_{a \in A} (Y_a \cap Z_a)$  where  $Y_s \cap Z_s$  is an open set in  $X_s$  for all  $s$  in the union of  $E$  and  $B$ . With  $E$  and  $B$  both being finite we have their union finite. Further  $Y_a \cap Z_a = X_a$  for all " $a$ " in  $A$  but not in either  $E$  or  $B$ . Notice now by definition we have  $\prod_{a \in A} (Y_a \cap Z_a)$  an element of  $\sigma^-$ .

Lemma 2': Given the non-empty product space  $\prod_{a \in A} X_a$  and  $F = \{F_a\}_{a \in A}$  a collection of closed subsets of the product space satisfying the finite intersection property then there exists a collection  $F'$  of subsets of the product space such that  $F'$  contains  $F$ ,  $F'$  satisfies the finite intersection property and  $F'$  is not a proper subcollection of any other collection having the first two properties.

To prove the validity of our lemma let  $\Omega = \{F_b\}$  be the family of all collections of subsets, not necessarily closed, of  $\prod_{a \in A} X_a$  such that  $F_b$  satisfies the finite intersection property. Notice  $\Omega$  is non-void since  $F$  belongs to  $\Omega$ . Partially-order  $\Omega$  by defining  $F_a \subset F_b$  to mean every set in  $F_a$  is a set in  $F_b$  but not conversely. The single collection  $F$  is a trivial simply-ordered subfamily of  $\Omega$ . Hence the maximality principle states there exists a maximal simply-ordered subfamily  $\Omega'$  of  $\Omega$  containing  $F$ . The desired collection  $F'$  is seen to be the largest element of  $\Omega'$ , which we must prove exists.

Let  $G^*$  be the union of all elements of  $\Omega'$ . We show  $G^*$  is in  $\Omega'$ , whence  $G^*$  is certainly the largest element of  $\Omega'$ . Also if  $G^*$  is in  $\Omega$  then  $G^*$  is not a proper subset of any other element of  $\Omega$ , for any

element containing  $G^*$  would be in  $\Omega'$  since such an element would be comparable to every element of  $\Omega'$ .

Suppose  $C_1, C_2, \dots, C_n$  are sets in  $G^*$ . For every  $j$  such that  $1 \leq j \leq n$  it is true that  $C_j$  is a set in some collection  $F_j$  in  $\Omega'$ . Since  $\Omega'$  is simply-ordered, some one of these, say  $F_k$ , contains all the others and hence contains all the sets  $C_1, C_2, \dots, C_n$ . Then, since  $F$  satisfies the finite intersection property we have  $\bigcap_{i=1}^n C_i \neq \emptyset$ . It follows that  $G^*$  satisfies the finite intersection hypothesis and is in  $\Omega$ . Also  $G^*$  is comparable with every element of  $\Omega'$ , so it is in  $\Omega'$ . Thus  $G^*$  is the  $F'$  desired.

We are now ready to state and demonstrate the proof of:

Theorem 2 The following are equivalent:

- (1)' The Tychonoff Theorem: Given a collection of non-empty spaces  $\{X_a\}_{a \in A}$ , the product space  $\prod_{a \in A} X_a$  will be compact if and only if each  $X_a$  is compact.
- (2)' Given a collection of non-empty compact spaces  $\{X_a\}_{a \in A}$ , we let  $\underline{S}$  be the family of all subsets of the Cartesian product which, for some  $a'$  in  $A$  and some  $U$  open in  $X_{a'}$ , are the set of all points  $\{x_a\}_{a \in A}$  with  $x_{a'} \in U$ . Then every covering of the product space by members of  $\underline{S}$  has a finite subcovering.
- (3)' The axiom of choice.

In this chapter we feel free to use any of the equivalent forms of the axiom of choice listed in Theorem 1. The proof of the Tychonoff Theorem is taken from Hall and Spencer [2], pp. 283-288. The actual equivalences in Theorem 2 are due to Kelley

(1)'  $\rightarrow$  (2)'

The implication of (2)' by (1)' is completely trivial since any open covering of the product space will have a finite subcovering.

(2)'  $\longrightarrow$  (3)'

The implication of the axiom of choice by (2)' is more interesting. Let  $\{X_a\}_{a \in A}$  be a collection of non-empty sets. We prove that the product space is non-empty. Define for every "a" in A  $Z_a = X_a \cup \{\Lambda\}$ , where  $\Lambda$  is any arbitrary symbol. Assign a topology to  $X_a$  by letting the empty set and complements of finite sets to be open. Topologize  $Z_a$  by saying a set is open in  $Z_a$  if it is open in  $X_a$  or if it is the union of an open set in  $X_a$  and  $\{\Lambda\}$ . It is evident with this topology that each  $Z_a$  is compact. Further, since the empty set is open in  $X_a$  we have  $\phi \cup \{\Lambda\} = \{\Lambda\}$  open in  $Z_a$  for every "a" in A.

Define  $\mathbb{I}$  as the set of all  $T_r$  where r is in A and  $T_r = \prod_{a \in A} Y_a$  such that  $Y_r$  is the point set  $\{\Lambda\}$  and  $Y_a = Z_a$  for "a" not equal to r. Now  $\mathbb{I}$  covers  $\prod_{a \in A} Z_a$  if  $\prod_{a \in A} X_a$  is empty. Applying (2)', there then exists a finite cover with say  $T = \{T_{r_1}, T_{r_2}, T_{r_3}, \dots, T_{r_n}\}$ .

Define  $P_{r_1}(s)$  as the  $r_1$ -th coordinate of the point s contained in  $\prod_{a \in A} Z_a$ . Notice now that we can define a point in the product space  $\prod_{a \in A} Z_a$  which is not covered by members of T (whence it must be true that  $\prod_{a \in A} X_a$  is not empty). In particular define the point  $s'$  such that  $P_a(s') = \Lambda$  for "a" not equal to  $r_1, r_2, r_3, \dots$ , or  $r_n$  and "a" contained in A and let  $P_a(s') = x_a$  where  $x_a$  is an element of  $X_a$  for "a" equal to  $r_1, r_2, r_3, \dots$ , or  $r_n$ .

(3)'  $\longrightarrow$  (1)'

We now prove the 'necessary' portion of the Tychonoff Theorem using the axiom of choice. That is, we show that given the collection of non-empty spaces  $\{X_a\}_{a \in A}$  that the individual spaces will be compact if the product space is compact. For every b in A define a function  $p_b$  on  $\prod_{a \in A} X_a$  into  $X_b$  as follows: For every point  $\{x_a\}_{a \in A}$  in the

product space  $p_b [\{x_a\}_{a \in A}] = x_b$ .  $p_b$  then, yields the  $b$ -th coordinate of any point in the product space. We assert  $p_b$  is a continuous mapping.

This is evident, for let  $O_b$  be any open set in  $X_b$  such that  $\{x_a\}_{a \in A}$  is an arbitrary point of  $p_b^{-1}(O_b)$ . Note that  $p_b^{-1}(O_b)$  is the set of all points of the product space such that the  $b$ -th coordinate is an element of  $O_b$ . This is precisely  $\prod_{a \in A} Y_a$  where  $Y_a = X_a$  for  $a \neq b$  and  $Y_b = O_b$  for  $O_b$  open in  $X$ . This, by definition is an open set. Thus  $p_b$  is a continuous mapping. Moreover  $p_b$  is a mapping onto  $X_b$  since (using the axiom of choice) given  $r$  an element of  $X_b$  it is true there exists a point of the product space having  $r$  as its  $b$ -th coordinate. Thus for every  $b$  in  $A$  we have  $X_b$  as the continuous image of the compact space  $\prod_{a \in A} X_a$ .  $X_b$  is itself then compact.

Conversely, now, we show that if each  $X_a$  of the collection  $\{X_a\}_{a \in A}$  is compact then the product space  $\prod_{a \in A} X_a$  is compact. The criterion which we use is a well known theorem from topology: A space is compact if and only if every family of closed sets having the finite intersection property has the intersection property. The reader is referred to Kelley, [5] p.136.

Let  $\{F_r\}_{r \in \Sigma}$  (generally  $\Sigma$  is different from  $A$ ) be a collection of closed subsets of the product space  $\prod_{a \in A} X_a$  having the finite intersection property.

Lemma (2)' asserts then that there exists a collection  $G = \{G_d\}_{d \in A}$  of subsets, not necessarily all closed, of  $\prod_{a \in A} X_a$  which contains the collection  $\{F_r\}_{r \in \Sigma}$  as a subcollection and which has the finite intersection property and no collection of subsets of the product space having  $G$  as a proper subcollection also has the finite intersection property. The collection  $G$  must therefore satisfy the follow-



ing conditions: (i) the intersection of any non-empty finite collection of sets which are elements of  $G$  is an element of  $G$ ; (ii) if  $L$  is a subset of the product space such that  $L$  intersects every element of  $G$ , then  $L$  is an element of  $G$ .

To show (i) is true let  $G'$  be a finite collection of sets which are elements of  $G$  - i.e.  $G' = \{G_e\}_{e \in E}$  for  $E$  a finite subset of  $\mathcal{A}$ . Consider  $\bigcap_{e \in E} G_e$ . This intersection is obviously not void since this is a finite collection of subsets of  $G$ . Further the collection composed of the sets of  $G$  (i.e.  $\{G_d\}_{d \in \mathcal{A}}$ ) and the set  $\bigcap_{e \in E} G_e$  is easily verified to have the finite intersection property. This last mentioned collection contains  $G$  as a proper subcollection if  $\bigcap_{e \in E} G_e$  is not in  $G$ . This is a contradiction to the results of lemma (2)'; therefore  $\bigcap_{e \in E} G_e$  must itself be an element of  $G$ .

To show (ii) is true let  $E$  be a finite subset of  $\mathcal{A}$  and let  $G' = \{G_e\}_{e \in E}$ . Now  $\bigcap_{e \in E} G_e$  is non-void, and is an element of  $G$  by (i); therefore, by hypothesis  $L \cap \bigcap_{e \in E} G_e$  is non-void. Clearly, then, this says that the collection of sets composed of  $L$  and  $\{G_d\}_{d \in \mathcal{A}}$  has the finite intersection property. Further, if  $L$  is not in  $G$  then this collection contains  $G$  as a proper subcollection. This is a contraction to the results of lemma (2)'; therefore,  $L$  must be an element of  $G$ .

Now, for each  $b$  contained in  $A$  denote  $X_b$  as the collection of closed sets  $\{\overline{p_b(G_d)}\}_{d \in \mathcal{A}}$ . We show that  $X_b$  is a family of closed subsets of  $X_b$  which has the finite intersection property. For each  $d$  contained in  $\mathcal{A}$  we have  $G_d$  contained in the product space  $\prod_{a \in \mathcal{A}} X_a$  and  $p_b$  maps the product space onto  $X_b$  for every  $b$  and also the closure of  $p_b(G_d)$  is a subset of  $X_b$ . Suppose now that we have the finite collection

$\{\overline{p_b(G_{d_i})}\}_{i=1,2,\dots,n}$  contained in  $\{\overline{p_b(G_d)}\}_{d \in \Lambda}$ . Then we have

$$\overline{p_b(G_{d_1})} \cap \overline{p_b(G_{d_2})} \cap \dots \cap \overline{p_b(G_{d_n})} \supseteq \bigcap_{i=1}^n p_b(G_{d_i}) \supseteq p_b(G_{d_1} \cap G_{d_2} \cap \dots \cap G_{d_n}) \neq \emptyset.$$

Thus for every  $b$  in the set  $\Lambda$  there exists, since  $X_b$  is compact, a point  $x_b$  in the intersection  $\bigcap_{d \in \Lambda} \overline{p_b(G_d)}$ . Here we use the fact that if a set is compact then any collection of non-empty closed sets which has the finite intersection property will have the intersection property. Let  $z = \{x_b\}_{b \in \Lambda}$  denote the point of the product space  $\prod_{a \in \Lambda} X_a$  thus chosen and let  $U$  be any open set in  $\sigma$ , the basis for the topology of the product space, that contains the point  $z$ . We shall show that  $U \cap F_r \neq \emptyset$  for every  $r$  contained in  $\Sigma$ . Since  $U$  is any open subset of the product space in  $\sigma$  containing  $z$ , it will then follow that  $z$  is a limit point of the closed set  $F_r$  for every  $r$  contained in  $\Sigma$ . Thus  $z$  is an element of  $\bigcap_{r \in \Sigma} F_r$  and our proof will be complete. Notice that if  $U'$  is an arbitrary open set containing the point  $z$  then there exists a set  $U$  in  $\sigma$  such that  $U$  is contained in  $U'$  and  $U$  contains  $z$ . Thus there is no loss of generality by allowing  $U$  to be an open set in  $\sigma$ .

We assert that  $p_b$  is an open mapping for every  $b$  in  $\Lambda$ . This is evident, for let  $\prod_{a \in \Lambda} Y_a$  be any open subset of  $\bigcap_{a \in \Lambda} X_a$ . Either  $Y_b = X_b$  or  $Y_b$  is an open subset of  $X_b$ . For each point  $\{y_a\}_{a \in \Lambda}$  of  $\prod_{a \in \Lambda} Y_a$  we have  $p_b(\{y_a\}_{a \in \Lambda}) = y_b$ . Further  $p_b$  is a mapping onto  $Y_b$  since (using the axiom of choice) given  $r$  in  $Y_b$  there is a point  $\{y'_a\}_{a \in \Lambda}$  such that  $y'_b = r$ . Thus the image of any open set  $\bigcap_{a \in \Lambda} Y_a$  is an open set. Hence,  $p_b$  is an open mapping, since every open set in  $\prod_{a \in \Lambda} X_a$  is the union of sets in  $\sigma$ .

Now, to see that  $U \cap F_r$  is not empty for every  $r$  in  $\Sigma$  notice first that for every  $d$  in  $\Lambda$  and  $b$  in  $\Lambda$  we have  $p_b(U) \cap \overline{p_b(G_d)}$  non-empty.

This is evident since  $z$  is an element of  $U$  and  $z = \{x_b\}_{b \in A}$  where  $x_b$  is contained in the intersection  $\bigcap_{d \in \Lambda} \overline{p_b(G_d)}$ . Since  $p_b$ , for every  $b$  in  $A$  is an open mapping, it then follows that the intersection of  $p_b(U)$  and  $p_b(G_d)$  is not empty for all  $d$  in  $\Lambda$  and all  $b$  in  $A$ . To verify this we know that  $p_b(U) \cap \overline{p_b(G_d)}$  is non-void; therefore, let  $z'$  be a point of the intersection. Now  $U$  is an open set and  $p_b$  an open mapping which then yields  $z'$  a point in the open set  $p_b(U)$  and also a point of the closure of the set  $p_b(G_d)$ . Having the latter property any open set containing  $z'$  will intersect  $p_b(G_d)$ , in particular  $p_b(U) \cap p_b(G_d) \neq \emptyset$ .

Consequently  $[p_b^{-1}(p_b(U))] \cap G_d$  is not empty for all  $b$  in  $A$  and all  $d$  in  $\Lambda$ . It is a consequence of this and property (ii) that  $p_b^{-1}(p_b(U))$  is an element of  $G$  for every  $b$  in  $A$ . But  $U = \bigcap_{b \in \Gamma} p_b^{-1}(p_b(U))$  where  $\Gamma$  is some finite subset of  $A$ . Consequently, by property (i),  $U$  is an element of  $G$ . Therefore  $U \cap F_r$  is non-void for all  $r$  in  $\Sigma$  since each  $F_r$  is an element of  $G$ .

CHAPTER IV  
TRICHOTOMY FOR CARDINAL NUMBERS

We now turn our attention to cardinal numbers.

Theorem 3 The axiom of choice and the law of trichotomy for cardinal numbers are equivalent.

The proof is due to Sierpinski [9], pp. 264-268, 407-409.

To show the axiom of choice, in the equivalent form of the well-ordering theorem, implies the law of trichotomy for cardinal numbers we first prove several necessary lemmas. Lemma 1: Let  $A$  be a non-empty set well-ordered by  $\alpha$ . If  $f(x)$  is a function, defined for the elements of  $A$ , such that  $f(x)$  is an element of  $A$  for  $x$  in  $A$  and  $f(x) \alpha f(y)$  for  $x$  and  $y$  contained in  $A$  with  $x \alpha y$ , then  $f(a) \alpha a$  can hold for no element " $a$ " of the set  $A$ . To show this is true, suppose that, for a certain element " $a$ " of the set  $A$ , we have the relation  $f(a) \alpha a$ . Let us denote by  $B$  the set of all those elements  $x$  of the set  $A$  for which  $f(x) \alpha x$ . By definition of the set  $B$  we have " $a$ " an element of  $B$  and the set  $B$ , as a non-empty subset of the well-ordered set  $A$ , has a first element,  $a_1$ . Since  $a_1$  is an element of  $B$  we have the relation  $f(a_1) \alpha a_1$ . Let  $a_2 = f(a_1)$ ;  $a_2$  will be an element of the set  $A$  and we have  $a_2 \alpha a_1$ , which, in view of the properties of the function  $f$ , implies the relation  $f(a_2) \alpha f(a_1)$ , i.e.  $f(a_2) \alpha a_2$ , which says that  $a_2$  is an element of  $B$ . But this is impossible since  $a_2 \alpha a_1$  and  $a_1$  is the first element of the set  $B$ . Thus the assumption that the relation  $f(a) \alpha a$  holds for a certain element " $a$ " of the set  $A$  leads to a

contradiction. This then proves the first lemma.

Lemma 2: A well-ordered set can be similarly mapped onto itself only identically.

Suppose that a well-ordered set  $A$  is similarly mapped onto itself; let  $f$  denote a function establishing that mapping,  $f^{-1}$  its inverse function. Of course the function  $f^{-1}$  also establishes a similar mapping of the set  $A$  onto itself. In virtue of the above remark neither  $f(a) \prec a$  nor  $f^{-1}(a) \prec a$  can hold for any element " $a$ " of the set  $A$ ; hence it follows also that we cannot have  $a \prec f(a)$ , for it would imply that  $f^{-1}(a) \prec f^{-1}(f(a))$ , i.e.  $f^{-1}(a) \prec a$ , since the function  $f$  establishes a similar mapping of the set  $A$  onto itself, which is impossible. Thus for each element " $a$ " of the set  $A$  both the relation  $f(a) \prec a$  and the relation  $a \prec f(a)$  are false, which proves that we must have  $f(a) = a$  for " $a$ " an element of  $A$ . Therefore we have our conclusion.

Lemma 3: Two similar well-ordered sets can be similarly mapped onto each other in one way only.

Suppose now that  $f$  and  $g$  are two different similar mappings of a well-ordered set  $A$  onto a set  $B$ . The function  $g^{-1}(f(a))$  obviously establishes a similar mapping of the set  $A$  onto itself and therefore it follows from the above discussion that  $g^{-1}(f(a)) = a$  for " $a$ " an element of  $A$ , whence  $f(a) = g(a)$  for " $a$ " in  $A$ , which contradicts the assumption that the mappings  $f$  and  $g$  are different from each other.

Lemma 4: If, for each segment of a well-ordered set  $A$ , different from  $A$ , there exists a similar segment of a well-ordered set  $A'$  different from  $A'$  and vice-versa, then the sets  $A'$  and  $A$  are similar. To show this, let " $a$ " be an arbitrary element of  $A$ . Define  $A_a$  as the set

of all elements of  $A$  that precede " $a$ ". By assumption there exists a similar segment  $S'_a$  of  $A'$  and as we know, only one such segment. The element  $a'$  of the set  $A'$  is thus well-defined by the element " $a$ " of the set  $A$ . Let  $f(a)=a'$ . The function  $f$  establishes a similar mapping of the set  $A$  onto the set  $A'$ .

Indeed, let  $a'$  be an arbitrary element of  $A'$  and corresponding to the segment  $A'_{a'}$  of the set  $A'$  corresponds, in virtue of our assumption, a similar segment of  $A_a$  of the set  $A$  and it follows from the definition of the function  $f$  that  $f(a)=a'$ . Thus each element of the set  $A'$  is an image of a certain element of the set  $A$ .

Finally if  $a_1$  and  $a_2$  are elements of  $A$  such that  $a_1 \prec a_2$ , then clearly  $A_{a_1}$  is similar to a proper segment of  $A_{a_2}$  and for  $a'_1=f(a_1)$  and  $a'_2=f(a_2)$  we have  $A'_{a'_1}$  similar to a proper segment of  $A'_{a'_2}$  since  $A_{a_1}$  is similar to  $A'_{a'_1}$  and  $A'_{a'_2}$  is similar to  $A_{a_2}$ . Therefore  $a'_1 \prec a'_2$ , hence each element of  $A'$  is an image of only one element of the set  $A$  and  $f$  establishes the similarity of the sets  $A$  and  $A'$ .

Now let  $A$  and  $A'$  be two given well-ordered sets such that in  $A$  there exists a segment different from  $A$  that is not similar to any segment of  $A'$  different from  $A'$ . Let  $A_a$  denote the smallest such segment of  $A$ . (From our discussion of segments in Chapter I it is clear that every <sup>proper</sup> segment in a well-ordered set  $A$  is of the form  $A_{a^*}$  for some  $a^*$  in  $A$ .) We shall prove that each segment of  $A'$  different from  $A'$  is similar to a segment of the set  $A_a$  different from  $A_a$ .

If there exists segments of  $A'$  different from  $A'$  not similar to any segment of  $A_a$  different from  $A_a$ , then let  $A'_{a'}$  denote the smallest

of them. Thus each segment  $A'_{b'}$  of the set  $A'_{a'}$  different from  $A'_{a'}$ , being a segment of the set  $A'_{a'}$  and being smaller than  $A'_{a'}$ , is similar to a segment  $A_b$  of the set  $A_a$  different from  $A_a$  and we have  $A_b \subset A_a$ . Therefore each segment of  $A'_{a'}$  different from  $A'_{a'}$  is similar to a segment of  $A_a$  different from  $A_a$ . But also each segment  $A_{a_1}$  of the set  $A_a$  different from  $A_a$  is, according to the definition of the segment  $A_a$ , similar to a segment  $A'_{a'_1}$  of the set  $A'$  different from  $A'$  and it follows from the definition of the segment  $A'_{a'_1}$  that  $A'_{a'_1} \subset A'_{a'}$ . Thus each segment of the set  $A_a$  different from  $A_a$  is similar to a segment of the set  $A'_{a'}$  different from  $A'_{a'}$ . From our discussion on page 31  $A_a$  and  $A'_{a'}$  are similar which contradicts the definition of  $A_a$ .

Thus we have proved that each segment of the set  $A'$  different from  $A'$  is similar to a segment of the set  $A_a$  different from  $A_a$ . By the definition of the segment  $A_a$  it follows that also each segment of the set  $A_a$  different from  $A_a$ , hence smaller than  $A_a$ , is similar to a segment of the set  $A'$ , different from  $A'$ .

Again by a previous discussion we have  $A'$  and  $A_a$  similar, hence the set  $A'$  is similar to a certain segment of  $A$  different from  $A$ .

We have proved that if  $A$  and  $A'$  are two well-ordered sets and if in the set  $A$  there exists a segment, different from  $A$ , that is not similar to any segment of the set  $A'$  different from  $A'$ , then the set  $A'$  is similar to a certain segment of the set  $A$  different from  $A$ .

It follows that if on the set  $A'$  there exists a segment different from  $A'$ , not similar to any segment of the set  $A$  different from  $A$ , then the set  $A$  is similar to a certain segment of the set  $A'$  different from  $A'$ .

Lemma 5: (Bernstein's Equivalence-Theorem) If  $M$  and  $N$  are two sets such that each is equivalent to a subset of the other, then  $M$  is equivalent (written  $M \sim N$ ) to  $N$ .

The proof is due to E. Kamke [4], pp. 22-25.

We first show the following proposition: (P) If  $M$  is equivalent to a subset  $M_2$ , then  $M$  is equivalent to every set  $M_1$  with the property that  $M_2 \subseteq M_1 \subseteq M$ .

Assume  $M_2 \subset M_1 \subset M$  with  $M$  equivalent to  $M_2$ . For convenience set  $M_2 = A$ ,  $M_1 - M_2 = B$  and  $M - M_1 = C$ . Then proposition (P) reads as follows:

(P\*) If  $A, B$ , and  $C$  are disjoint sets and  $A \cup B \cup C$  is equivalent to  $A$  then  $A \cup B \cup C$  is equivalent to  $A \cup B$ .

Now, according to the hypothesis that  $A \cup B \cup C$  is equivalent to  $A$ , there exists a mapping  $\mathcal{Q}$  of the set  $A \cup B \cup C$  on  $A$ . Let  $A_1$ ,  $B_1$ , and  $C_1$  be the subsets of  $A$  which are the images of  $A$ ,  $B$ , and  $C$  respectively. Then (1a)  $A_1 \cup B_1 \cup C_1 = A$  and (1b)  $A \sim A_1$ ,  $B \sim B_1$ , and  $C \sim C_1$ . Further  $A_1$ ,  $B_1$ , and  $C_1$  are disjoint.

Since  $A$  is mapped by  $\mathcal{Q}$  on  $A_1$ , it follows from (1a) that the subsets  $A_1$ ,  $B_1$ , and  $C_1$  of  $A$  are mapped by  $\mathcal{Q}$  on subsets  $A_2$ ,  $B_2$ , and  $C_2$  respectively, of  $A_1$ . Then we have (2a)  $A_2 \cup B_2 \cup C_2 = A_1$  and (2b)  $A_1 \sim A_2$ ,  $B_1 \sim B_2$ , and  $C_1 \sim C_2$  and further  $A_2$ ,  $B_2$ , and  $C_2$  are disjoint. The next step leads to three sets  $A_3$ ,  $B_3$ , and  $C_3$  with (3a)  $A_3 \cup B_3 \cup C_3 = A_2$  and (3b)  $A_2 \sim A_3$ ,  $B_2 \sim B_3$ ,  $C_2 \sim C_3$ . Further,  $A_3$ ,  $B_3$ , and  $C_3$  are disjoint; et cetera.

Due to  $A_1 \sim A_1 \sim A_2 \sim \dots$ , the process does not terminate.

Note especially the equivalence (I)  $C \sim C_1 \sim C_2 \sim \dots$  arising from



(1b), (2b), ... .

If we set  $D = A_1 \cap A_2 \cap A_3 \cap \dots$ , which may be void, we have

$$A \cup B \cup C = D \cup B \cup C \cup B_1 \cup C_1 \cup B_2 \cup C_2 \cup \dots$$

$$A \cup B = D \cup B \cup C_1 \cup B_1 \cup C_2 \cup B_2 \cup C_3 \cup \dots$$

Here, on the right-hand side of the first equation, all the terms are mutually exclusive, and the same is true of the second equation. Hence,  $A \cup B$  is mapped on  $A \cup B \cup C$  and (P\*) is proved if we succeed in establishing a mapping between every term on the right-hand side of the first equation and the term directly below it in the second equation. The existence of this mapping, however, is ensured by (I).

Now, to prove our lemma, let  $M_1$  and  $N_1$  be subsets of  $M$  and  $N$  respectively such that  $M \sim N_1$  and  $N \sim M_1$ . By means of a mapping resulting from the last equivalence, the set  $N$  is mapped onto  $M_1$  and hence, in particular, the subset  $N_1$  is mapped on a subset,  $M_2$ , of  $M$ . Thus  $M_2 \subseteq M_1 \subseteq M$  and  $M \sim N_1 \sim M_2$ .

Consequently by (P),  $M_1 \sim M$ ; and since  $M_1 \sim N$  we have also  $M \sim N$ .

Notice that the axiom of choice (or any of its equivalent forms) has not been used in the proof of any of the preceding five lemmas.

Thus two well-ordered sets are either similar or one and only one of them is similar to a certain segment of the other set, different from that set.

As a direct consequence then, always one and only one of the formulae  $\bar{A} = \bar{A}'$ ,  $\bar{A} < \bar{A}'$ ,  $\bar{A} > \bar{A}'$  holds for  $A$  and  $A'$  being well-ordered sets.

It is an easy task to now show that the axiom of choice implies the law of trichotomy. Let  $A_1$  and  $A_2$  be two arbitrary sets. By the well ordering theorem, or equivalently the axiom of choice,  $A_1$  and  $A_2$  can be well-ordered, and by our previous discussion the law of trichotomy is immediate.

The converse, i.e. the implication of the axiom of choice by the law of trichotomy, requires perhaps as much development as the above implication did.

Let  $M$  denote a set and  $U(M)$  the set of all subsets of  $M$  and  $UU(M)$  the set of all sets whose elements are subsets of the set  $M$ . Notice  $UU(M)$  is the set of all subsets of  $U(M)$ . By  $UUU(M)$ , then, we mean the set of all subsets of  $UU(M)$ . We first assert that we are able to define a function  $f(M)$  which associates with every set  $M$  a certain well-ordered set  $f(M)$  contained in  $UUU(M)$ , whose power is neither equal to nor less than the power of the set  $M$ .

Let  $F$  denote the set of all those sets of the subsets of  $M$  which are well-ordered according to the relation  $<$ , between the subsets of the set  $M$  which are their elements. This for example if  $M = \{1, 2, 3, \dots\}$  then  $\{\{3\}\}$  is an element of  $F$ ,  $\{\{1, 2\}, \{1, 2, 5\}\}$  is an element of  $F$  but  $\{\{1\}, \{2, 3\}\}$  is not an element of  $F$ . Let us divide the sets belonging to the set  $F$  into classes, assigning two

sets to the same class if and only if they are similarly well-ordered and let  $\phi$  denote the set of classes obtained in this manner. The classes are of course subsets of the set  $F$ , hence it follows that the set  $\phi$  is a subset of the set  $U(F)$  and since  $F$  is a subset of  $UU(M)$  we have  $U(F)$  a subset of  $UUU(M)$  and thus  $\phi$  is a subset of  $UUU(M)$ .

Before continuing our proof we prove the following lemma: The set of all proper segments of a well-ordered set is well-ordered by inclusion.

Let  $S$  be a well-ordered set. Now, let  $S^*$  be a collection of proper segments of  $S$ . If  $S^*$  contains the proper segment  $A$ , then  $A$  has the form  $A_a$  for some " $a$ " in  $S$ . Define  $K$  as the set of all " $a$ " such that  $A_a$  is a segment in the collection  $S^*$ .  $S$  is well-ordered; therefore,  $K$  has a first element, say  $a'$ . Now if  $A_{a_1}$  is in  $S^*$  we have  $a_1$  in  $K$  such that  $a'$  precedes  $a_1$ . We then immediately have  $A_{a'} \subset A_{a_1}$ . Thus  $S^*$ , being any collection of proper segments of  $S$ , has a first element when ordered by inclusion.

Now, suppose that  $K_1$  and  $K_2$  are two different classes belonging to  $\phi$  with  $A_1$  an element of  $K_1$  and  $A_2$  an element of  $K_2$ , then  $A_1$  and  $A_2$  are well-ordered but they are not similar since they belong to different classes. Thus by an earlier discussion one of these sets is similar to a proper segment of the other. Let  $K_1 < K_2$  if  $A_1$  is similar to a proper segment of  $A_2$ . If, instead we chose  $A'_1$  and  $A'_2$  such that  $A'_1$  is an element of  $K_1$  and  $A'_2$  is an element of  $K_2$  then we should have  $A'_1$  similar to a proper segment of  $A'_2$ . This is evident, for suppose  $A'_2$  were similar to a proper segment of  $A'_1$ .  $A_2$  and  $A'_2$  belong to the same class  $K_2$  in  $\phi$  and are therefore similar. Thus, if  $A'_2$  is

similar to a proper segment of  $A_1$  then  $A_2$  is similar to a proper segment of  $A_1$ . Extending this argument we obtain  $A_2$  similar to a proper segment of  $A_1$  which is impossible since we have assumed  $A_1$  similar to a proper segment of  $A_2$ .

We assert  $\phi$  is well-ordered by the relation  $<$ . To show this let  $\underline{K}$  be a subset of  $\phi$ . Choose  $K$  an element of  $\underline{K}$  and  $C$  an element of  $K$ .  $C$  is well-ordered by  $<$ . Define  $\underline{G}$  to be equal to the set of all  $G$  such that  $G$  is an element of  $\underline{K}$  and  $G \subset K$ . If  $K_1$  is in  $\underline{K}$  and  $B_1$  in  $K_1$  and  $K_1 < K$  then  $B_1$  is similar to one and only one segment of  $C$ . Thus  $K_1$  contains a segment of  $C$ , say  $C_1$ . Hence each element  $G$  of  $\underline{K}$  which is  $< K$  contains exactly one segment  $C_G$  of  $C$ . This segment  $C_G$  is of the form  $C_{a_G}$  for some  $a_G$  in  $C$ . Since, by our previous lemma, the collection of proper segments of  $C$  is well-ordered by  $<$ , the set of all  $C_G$  such that  $G$  is in  $\underline{G}$  has a first element, say  $C_{a_{G^*}}$ . Now  $G^*$  is the first element of  $\underline{G}$ , otherwise if there exists  $G'$  such that  $G' < G^*$  we would have  $C_{a_{G'}} \subset C_{a_{G^*}}$ , contradicting  $C_{a_{G^*}}$  being the first element of the collection of proper segments  $C_G$  well-ordered by  $<$ .

We prove  $\bar{\phi} \leq \bar{M}$  is impossible whence we have our conclusion by defining  $f(M) = \phi$ .

To this end assume  $\bar{\phi} \leq \bar{M}$ . This then says there exists a subset  $M_1$  of  $M$  such that  $\phi$  is in one-to-one correspondence with  $M_1$  and there exists a mapping of the set  $\phi$  onto  $M_1$  by which  $M_1$  is well-ordered. Thus  $\phi$  and  $M_1$  are similar.

Denote by  $A$  the set of all proper segments of  $M_1$ . 'A' will be a certain set of subsets of  $M_1$ , well-ordered by the relation  $<$ ; hence,  $A$  belongs to some set  $K$  of  $\phi$ . The mapping  $a \rightarrow A_a$  is a similar mapping

of  $M_1$  onto  $A$ .

Likewise if  $B$  is a collection of proper segments of  $A$  well-ordered by  $<$ ,  $B$  is similar to  $A$ . Each element of  $B$  belongs to some  $E_1$  with  $E_1 < E$  since  $A$  is itself an element of  $E$ . Also each  $E_1$  such that  $E_1 < E$  contains exactly one proper segment of  $A$ ; hence, exactly one element  $b_{E_1}$  of  $B$ . If  $E_2 < E_1 < E$  then  $b_{E_2} < b_{E_1}$ ; so the mapping  $E_1 \rightarrow b_{E_1}$  is a similarity between  $B$  and the segment of  $\phi$  determined by  $E$ . Thus this proper segment of  $\phi$  is similar to  $\phi$  which is a contradiction. Hence,  $\bar{\phi} \leq \bar{E}$  is false hence by trichotomy  $\bar{\phi} > \bar{E}$ .

Now let  $m$  denote a non-finite cardinal number and let  $M$  be a set of power  $m$ . The power of the set  $f(M)$  will obviously be an aleph, independent of the choice of the set  $M$  of power  $m$ ; let us denote it by  $\aleph'(m)$ .

In view of  $f(M)$  being contained in  $\mathcal{P}(M)$  and  $\bar{f} = m$  we have  $\aleph'(m) \leq 2^{2^m}$ , hence for every non-finite cardinal number  $m$  there exists an aleph  $\aleph'(m)$  such that  $\aleph'(m) \leq 2^{2^m}$  and  $\aleph'(m)$  is not less than or equal to  $m$ .

We conclude our proof by asserting the law of trichotomy implies the axiom of choice. To this end let  $M$  be a set and  $m$  its power. Then there exists an aleph  $\aleph'(m)$  which is not less than or equal to  $m$ . From the law of trichotomy we have  $m < \aleph'(m)$  which says  $M$  is equivalent to a subset of a well-ordered set of power  $\aleph'(m)$ , hence immediately there is a well-ordering relation on the set  $M$ .

## CHAPTER V

### APPLICATIONS

The extremes of views held by mathematicians on the axiom of choice are evident by those held by N. Lusin and Hilbert. Fraenkel and Bar-Hillel [1] ,pp56-60, suggest that Lusin considered the proof of any theorem by means of Zermelo's axiom as totally lacking in significance and value. The same authors also mention that according to Hilbert the axiom of choice is based on a general logical principle which is necessary and quite indispensable for the foundation of mathematical inference.

Why then use the axiom of choice when there is a question of its validity? Thus far the axiom of choice has not contradicted other accepted axioms. From this axiom mathematicians have drawn a great many conclusions, none of which so far has lead to a contradiction. Many of these conclusions have themselves been validated by proofs independent of the axiom of choice.

A strong argument in favor of Zermelo's axiom was made in 1938 when the mathematician K. Gödel proved that the axiom of choice is consistent with other generally accepted axioms provided these other axioms are consistent with one another.

Much research has been done in attempting to prove items without the aid of the axiom of choice but little or nothing has been accomplished in directly denying the axiom and attempting to draw further conclusions from this denial. Much can be done without directly applying the axiom of choice. In topology the axiom of choice is usually assumed from the start, while algebraists are inclined to proceed as

far as possible without it. In analysis, Landau developed the theory of real and complex numbers without the axiom of choice. Hardy developed calculus in a rigorous method, employing the axiom of choice in very few instances, each of which could have been avoided.

There are axiomatic systems of set theory, notably that of von Neumann, where there is no explicit appearance of the axiom of choice but it is a consequence of the other axioms.

We see then, in previous chapters, that there are equivalent statements of the axiom of choice in various branches of mathematics.

It may be instructional at this time to note some of the applications of the axiom of choice. Of course the axiom of choice can be used as a valuable heuristic tool - i.e. the axiom of choice can be used to discover conclusions for which we can then seek proofs which do not employ said axiom.

Sierpiński [9], pp 11-13, lists several theorems which he asserts mathematicians have been unable to prove without the axiom of choice.

A few of them follow:

- 1) If we decompose any set  $A$  into pairwise disjoint non-empty subsets, then the set of all those subsets is of power less than or equal to the power of the set  $A$ .
- 2) If we decompose a set of the power of the continuum into two subsets, then at least one of them will be of the power of the continuum.
- 3) In order that an infinite set be denumerable it is necessary and sufficient that it be equivalent to each of its infinite subsets. (Here, only the "sufficiency" requires the axiom of choice.)
- 4) Every infinite set is the sum of an infinite series of non-empty disjoint sets.
- 5) Every infinite set of real numbers contains a denumerable subset.

- 6) The sum of an infinite series of disjoint non-empty sets, finite or denumerable, is a denumerable set.

Again, Sierpiński, [8], relates the following:

**Analysis:** Nous dirons qu'une fonction  $f(x)$  définie dans un intervalle  $(a,b)$  est continue au point  $x_0$  de cet intervalle au sens de Cauchy, si pour tout nombre positif  $\epsilon$  un nombre positif  $\delta$  existe tel que l'inégalité  $|x-x_0| < \delta$  entraîne, pour tout nombre  $x$  de l'intervalle  $(a,b)$ , l'inégalité  $|f(x)-f(x_0)| < \epsilon$ . Nous dirons, d'autre part, qu'une fonction  $f(x)$  définie dans un intervalle  $(a,b)$ , est continue au point  $x_0$  de cet intervalle au sens de M. Heine si, pour toute suite infinie  $x_n$  de nombres de l'intervalle  $(a,b)$ , la formule (1)  $\lim_{n \rightarrow \infty} x_n = x_0$  entraîne la formule (2)  $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$ . On peut démontrer sans peine, sans l'aide de l'axiome de M. Zermelo, que, si une fonction  $f(x)$  définie dans un intervalle  $(a,b)$  est continue au point  $x_0$  de cet intervalle au sens de Cauchy, elle est aussi continue au point  $x_0$  au sens de M. Heine; mais la démonstration de la proposition réciproque s'appuie sur l'axiome de M. Zermelo.

#### Measure Theory:

Sans se baser sur l'axiome de M. Zermelo, on ne sait pas démontrer le théorème fondamental de mesure lebesguienne, théorème d'après lequel l'ensemble - somme d'une infinité dénombrable mesurables est un ensemble mesurable.

Currently mathematicians are unable to demonstrate the existence of a non-measurable set without the axiom of choice. Kolmogorev and Fomin, [7], p.14, exhibit an example of a non-measurable set constructed on the circumference of a circle. Their discussion follows:

Let  $C$  be a circumference of length 1, and let  $\alpha$  be an irrational number. Partition the points of  $C$  into classes by the following rule: two points of  $C$  belong



to the same class if and only if one can be carried into the other by a rotation of  $C$  through an angle  $n\alpha$  (degrees) where  $n$  is an integer. Each class is clearly countable. We now select a point from each class. We show that the resulting set  $\bar{\Phi}$  is non-measurable. Denote by  $\bar{\Phi}_n$  the set obtained by rotating  $\bar{\Phi}$  through the angle  $n\alpha$ . It is easily seen that all the sets  $\bar{\Phi}_n$  are pairwise disjoint and that their union is  $C$ . If the set  $\bar{\Phi}$  were measurable the sets  $\bar{\Phi}_n$  congruent to it would also be measurable. Since  $C = \bigcup_{n=-\infty}^{\infty} \bar{\Phi}_n$  the  $\sigma$ -additivity of the measure would imply that  $\bar{\Phi}_n \cap \bar{\Phi}_m = \emptyset$  for  $n \neq m$ . But congruent sets must have the same measure! (A)  $\sum_{n=-\infty}^{\infty} \mu(\bar{\Phi}_n) = 1$ . The last equality shows that (A) is impossible, since the sum of the series on the left side of (A) is zero if  $\mu(\bar{\Phi}) = 0$  and is infinity if  $\mu(\bar{\Phi})$  is greater than 0. Hence, the set  $\bar{\Phi}$  (and consequently every set  $\bar{\Phi}_n$ ) is nonmeasurable.

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